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Applications of Pumping Lemma for Context-Free Languages

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The Pumping Lemma:

- For infinite context-free language L

  there exists an integer m such that

  for any string $w \in L, \quad |w| \geq m$

  we can write $w = uvxyz$

  with lengths $|vxy| \leq m$ and $|vy| \geq 1$

  and it must be:

$$uv^ixy^iz \in L, \quad \text{for all } i \geq 0$$

Non-context free languages

$\{a^nb^nc^n : n \geq 0\}$ $\quad\quad$ $\{vv : v \in \{a,b\}\}$

Context-free languages

$\{a^nb^n : n \geq 0\}$ $\quad\quad$ $\{ww^R : w \in \{a,b\}^*\}$

Theorem: The language

$$L = \{a^nb^nc^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^nb^nc^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

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$$L = \{a^nb^nc^n : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string $w \in L$ with length $|w| \geq m$

We pick: $w = a^mb^mc^m$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

We can write: $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

We examine **all** the possible locations of string vxy in w

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Case 1: vxy is within a^m

m $\quad m$ $\quad m$

$aaa...aaa\ bbb...bbb\ ccc...ccc$

u $\quad vxy$ $\quad z$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Case 1: v and y consist from only a

m $\quad m$ $\quad m$

$aaa...aaa\ bbb...bbb\ ccc...ccc$

u $\quad vxy$ $\quad z$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$w = a^m b^m c^m$

$w = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Case 1: Repeating v and y

$k \geq 1$

$m+k$ $\quad m$ $\quad m$

$aaaaaa...aaaaaa\ bbb...bbb\ ccc...ccc$

u $\quad v^2 xy^2$ $\quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$k \geq 1$

$$\underbrace{a}_{u}\underbrace{aaaa...aaaa}_{v^2xy^2}\underbrace{a\ bbb...bbb\ ccc...ccc}_{z}$$

$$\overbrace{aaaaaa...aaaaaa}^{m+k}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$k \geq 1$

However: $uv^2xy^2z = a^{m+k} b^m c^m \notin L$

Contradiction!!!

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 2: vxy is within b^m

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\ b}_{u}\underbrace{bb...b}_{vxy}\underbrace{bb\ ccc...ccc}_{z}$$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 2: Similar analysis with case 1

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\ bbb...}_{u}\underbrace{bbb}_{vxy}\underbrace{ccc...ccc}_{z}$$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 3: vxy is within c^m

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\ bbb...bbb\ ccc...}_{u}\underbrace{cc}_{vxy}\underbrace{c}_{z}$$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 3: Similar analysis with case 1

$$\overbrace{aaa...aaa}^{m}\ \overbrace{bbb...bbb}^{m}\ \overbrace{ccc...ccc}^{m}$$

$$\underbrace{aaa...aaa\ bbb...bbb\ c}_{u}\underbrace{cc...c}_{vxy}\underbrace{cc}_{z}$$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: vxy overlaps a^m and b^m

$$\overbrace{aaa...aaa}^{m}\overbrace{bbb...bbb}^{m}\overbrace{ccc...ccc}^{m}$$

$u \quad vxy \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 1: v contains only a
y contains only b

$$\overbrace{aaa...aaa}^{m}\overbrace{bbb...bbb}^{m}\overbrace{ccc...ccc}^{m}$$

$u \quad vxy \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 1: v contains only a
y contains only b

$k_1 + k_2 \geq 1$

$$\overbrace{aaa...aaaaaaa}^{m+k_1}\overbrace{bbbbbbb...bbb}^{m+k_2}\overbrace{ccc...ccc}^{m}$$

$u \quad v^2xy^2 \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$k_1 + k_2 \geq 1$

$$\overbrace{aaa...aaaaaaa}^{m+k_1}\overbrace{bbbbbbb...bbb}^{m+k_2}\overbrace{ccc...ccc}^{m}$$

$u \quad v^2xy^2 \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$k_1 + k_2 \geq 1$

However: $uv^2xy^2z = a^{m+k_1} b^{m+k_2} c^m \notin L$

Contradiction!!!

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 2: v contains a and b
y contains only b

$$\overbrace{aaa...aaa}^{m}\overbrace{bbb...bbb}^{m}\overbrace{ccc...ccc}^{m}$$

$u \quad vxy \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 2: v contains a and b

$k_1 + k_2 + k \geq 1$ $\qquad$ y contains only b

$m \quad k_1 \quad k_2 \quad m+k \quad m$

$aaa...aaaaabbaabb\ bbbbbbb...bbb\ ccc...ccc$

$u \quad v^2xy^2 \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$k_1 + k_2 + k \geq 1$

$m \quad k_1 \quad k_2 \quad m+k \quad m$

$aaa...aaaaabbaabb\ bbbbbbb...bbb\ ccc...ccc$

$u \quad v^2xy^2 \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

However: $\qquad k_1 + k_2 + k \geq 1$

$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$

Contradiction!!!

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 3: v contains only a

y contains a and b

$m \quad m \quad m$

$aaa...aaa\ bbb...bbb\ ccc...ccc$

$u \quad vxy \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 4: Possibility 3: v contains only a

y contains a and b

Similar analysis with Possibility 2

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 5: vxy overlaps b^m and c^m

$m \quad m \quad m$

$aaa...aaa\ bbb...bbb\ ccc...ccc$

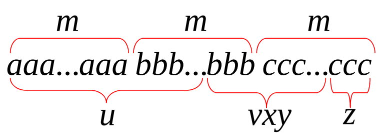
$u \quad vxy \quad z$

$L = \{a^n b^n c^n : n \geq 0\}$

$w = a^m b^m c^m$

$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

Case 5: Similar analysis with case 4



There are no other cases to consider

(since $|vxy| \leq m$, string vxy cannot overlap a^m, b^m and c^m at the same time)

In all cases we obtained a contradiction

Therefore: The original assumption that

$L = \{a^n b^n c^n : n \geq 0\}$

is context-free must be wrong

Conclusion: L is not context-free

Theorem: The language

$L = \{vv : v \in \{a,b\}^*\}$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$L = \{vv : v \in \{a,b\}^*\}$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$L = \{vv : v \in \{a,b\}^*\}$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

we pick: $a^m b^m a^m b^m \in L$

$$L = \{vv : v \in \{a,b\}^*\}$$

We can write: $a^m b^m a^m b^m = uvxyz$

with lengths $|vxy| \le m$ and $|vy| \ge 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \ge 0$

$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \quad |vxy| \le m \quad |vy| \ge 1$

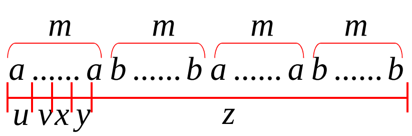
We examine **all** the possible locations of string vxy in $a^m b^m a^m b^m$

$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \quad |vxy| \le m \quad |vy| \ge 1$

Case 1: vxy is within the first a^m

$v = a^{k_1} \quad y = a^{k_2} \qquad k_1 + k_2 \ge 1$



$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \quad |vxy| \le m \quad |vy| \ge 1$

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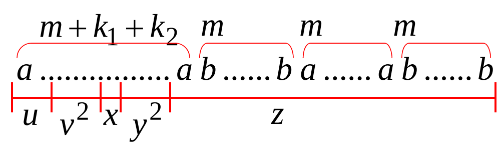


$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \quad |vxy| \le m \quad |vy| \ge 1$

Case 1: vxy is within the first a^m

$a^{m+k_1+k_2} b^m a^m b^m = uv^2 xy^2 z \quad \notin L$

$k_1 + k_2 \ge 1$

$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \quad |vxy| \le m \quad |vy| \ge 1$

Case 1: vxy is within the first a^m

$a^{m+k_1+k_2} b^m a^m b^m = uv^2 xy^2 z \quad \notin L$

However, from Pumping Lemma: $uv^2 xy^2 z \in L$

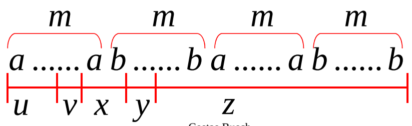
Contradiction!!!

$L = \{vv : v \in \{a,b\}^*\}$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 2: v is in the first a^m
y is in the first b^m

$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \ge 1$

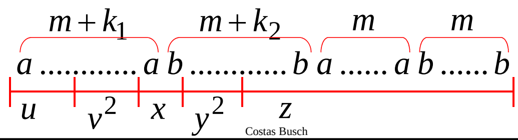


$L = \{vv : v \in \{a,b\}^*\}$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 2: v is in the first a^m
y is in the first b^m

$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \ge 1$



$L = \{vv : v \in \{a,b\}^*\}$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 2: v is in the first a^m
y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

$k_1 + k_2 \ge 1$

$L = \{vv : v \in \{a,b\}^*\}$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 2: v is in the first a^m
y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

However, from Pumping Lemma: $uv^2xy^2z \in L$

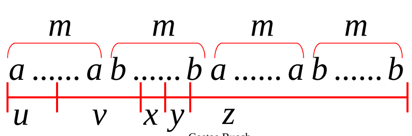
Contradiction!!!

$L = \{vv : v \in \{a,b\}^*\}$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \ge 1$

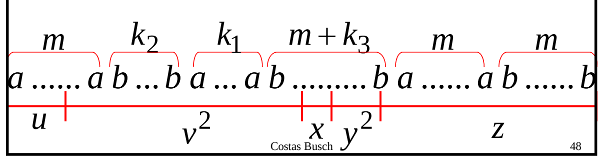


$L = \{vv : v \in \{a,b\}^*\}$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \ge 1$



$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{m+k_3} a^m b^m = uv^2 xy^2 z \quad \notin L$$

$k_1, k_2 \ge 1$

$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 3: v overlaps the first $a^m b^m$
y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{k_3} a^m b^m = uv^2 xy^2 z \quad \notin L$$

However, from Pumping Lemma: $uv^2 xy^2 z \in L$

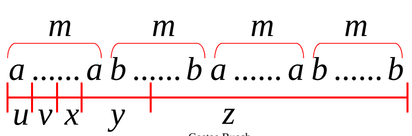
Contradiction!!!

$$L = \{vv : v \in \{a,b\}^*\}$$

$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$

Case 4: v in the first a^m
y Overlaps the first $a^m b^m$

Analysis is similar to case 3



Other cases: vxy is within $a^m \boxed{b^m} a^m b^m$

or

$a^m b^m \boxed{a^m} b^m$

or

$a^m b^m a^m \boxed{b^m}$

Analysis is similar to case 1:

$\boxed{a^m} b^m a^m b^m$

More cases: vxy overlaps $a^m \boxed{b^m a^m} b^m$

or

$a^m b^m \boxed{a^m b^m}$

Analysis is similar to cases 2,3,4:

$\boxed{a^m b^m} a^m b^m$

There are no other cases to consider

Since $|vxy| \le m$, it is impossible
vxy to overlap:

$\boxed{a^m b^m a^m} b^m$

nor

$a^m \boxed{b^m a^m b^m}$

nor

$\boxed{a^m b^m a^m b^m}$

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{vv : v \in \{a,b\}^*\}$$

is context-free must be wrong

Conclusion: L is not context-free

Non-context free languages

$\{a^n b^n c^n : n \geq 0\}$ $\{ww : w \in \{a,b\}\}$

$\{a^{n!} : n \geq 0\}$

Context-free languages

$\{a^n b^n : n \geq 0\}$ $\{ww^R : w \in \{a,b\}^*\}$

Theorem: The language

$$L = \{a^{n!} : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^{n!} : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^{n!} : n \geq 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

we pick: $a^{m!} \in L$

$$L = \{a^{n!} : n \geq 0\}$$

We can write: $a^{m!} = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

$L = \{a^{n!} : n \geq 0\}$

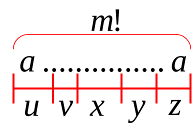
$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

We examine **all** the possible locations of string vxy in $a^{m!}$

There is only one case to consider

$L = \{a^{n!} : n \geq 0\}$

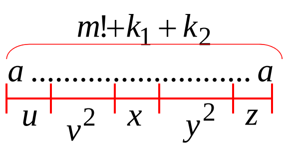
$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$



$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$

$L = \{a^{n!} : n \geq 0\}$

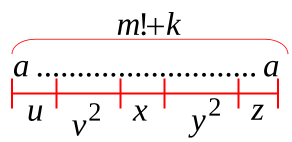
$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$



$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$

$L = \{a^{n!} : n \geq 0\}$

$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$



$k = k_1 + k_2$

$v = a^{k_1} \qquad y = a^{k_2} \qquad 1 \leq k \leq m$

$L = \{a^{n!} : n \geq 0\}$

$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

$$a^{m!+k} = uv^2xy^2z$$

$$1 \leq k \leq m$$

Since $1 \leq k \leq m$, for $m \geq 2$ we have:

$$\begin{aligned} m!+k &\leq m!+m \\ &< m!+m!m \\ &= m!(1+m) \\ &= (m+1)! \end{aligned}$$

$$m! < m!+k < (m+1)!$$

$$L = \{a^{n!} : n \geq 0\}$$

$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

$$m! < m! + k < (m+1)!$$

$$a^{m!+k} = uv^2xy^2z \quad \notin L$$

$$L = \{a^{n!} : n \geq 0\}$$

$a^{m!} = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$

However, from Pumping Lemma: $uv^2xy^2z \in L$

$$a^{m!+k} = uv^2xy^2z \quad \notin L$$

Contradiction!!!

We obtained a contradiction

Therefore: The original assumption that

$L = \{a^{n!} : n \geq 0\}$

is context-free must be wrong

Conclusion: L is not context-free

Non-context free languages

$\{a^n b^n c^n : n \geq 0\}$ $\qquad$ $\{ww : w \in \{a,b\}\}$

$\{a^{n^2} b^n : n \geq 0\}$ $\qquad$ $\{a^{n!} : n \geq 0\}$

Context-free languages

$\{a^n b^n : n \geq 0\}$ $\qquad$ $\{ww^R : w \in \{a,b\}^*\}$

Theorem: The language

$L = \{a^{n^2} b^n : n \geq 0\}$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^{n^2} b^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Since L is context-free and infinite we can apply the pumping lemma

$$L = \{a^{n^2} b^n : n \ge 0\}$$

Pumping Lemma gives a magic number m such that:

Pick any string of L with length at least m

we pick: $a^{m^2} b^m \in L$

$$L = \{a^{n^2} b^n : n \ge 0\}$$

We can write: $a^{m^2} b^m = uvxyz$

with lengths $|vxy| \le m$ and $|vy| \ge 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \ge 0$

$$L = \{a^{n^2} b^n : n \ge 0\}$$

$a^{m^2} b^m = uvxyz$ $\quad |vxy| \le m$ $\quad |vy| \ge 1$

We examine **all** the possible locations

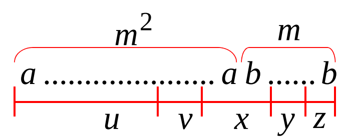
of string vxy in $a^{m^2} b^m$

$$L = \{a^{n^2} b^n : n \ge 0\}$$

$a^{m^2} b^m = uvxyz$ $\quad |vxy| \le m$ $\quad |vy| \ge 1$

Most complicated case: v is in a^m

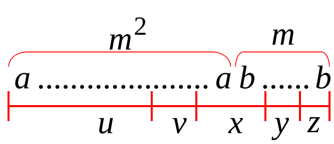
y is in b^m



$$L = \{a^{n^2} b^n : n \ge 0\}$$

$a^{m^2} b^m = uvxyz$ $\quad |vxy| \le m$ $\quad |vy| \ge 1$

$v = a^{k_1}$ $\quad y = b^{k_2}$ $\quad 1 \le k_1 + k_2 \le m$

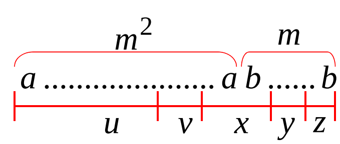


$$L = \{a^{n^2} b^n : n \ge 0\}$$

$a^{m^2} b^m = uvxyz$ $\quad |vxy| \le m$ $\quad |vy| \ge 1$

Most complicated sub-case: $k_1 \ne 0$ and $k_2 \ne 0$

$v = a^{k_1}$ $\quad y = b^{k_2}$ $\quad 1 \le k_1 + k_2 \le m$

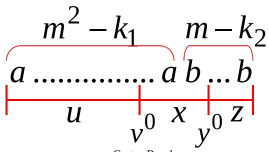


$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$



$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z$$

$k_1 \neq 0$ and $k_2 \neq 0$ $\qquad 1 \leq k_1 + k_2 \leq m$

⇓

$$(m - k_2)^2 \leq (m-1)^2$$
$$= m^2 - 2m + 1$$
$$< m^2 - k_1$$

⇓

$$m^2 - k_1 \neq (m - k_2)^2$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$m^2 - k_1 \neq (m - k_2)^2$$

⇓

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z \quad \notin L$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

However, from Pumping Lemma: $uv^0 xy^0 z \quad \in L$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z \quad \notin L$$

Contradiction!!!

When we examine the rest of the cases
we also obtain a contradiction

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^{n^2} b^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free